The Mathematics Department Presents
The Problem of the Month
September 2003
Cubes Inside Pyramids

The Problem:

Consider a pyramid with a square base, 2 units on a side. Suppose that the height of the pyramid is 1 unit. Imagine placing a cube inside this pyramid such that the base of the cube is flush with the base of the pyramid and such that each of the 4 top corners of the cube is touching each of the 4 slanted edges of the pyramid. If the length of each of the 12 edges of the cube equals some value, E, what is E? Justify your answer.

A Solution: Drop a perpendicular from the vertex of the pyramid to its base. From the point that this perpendicular touches the base, construct a line orthogonal to any one of the sides.
From the given dimensions of the pyramid, it is evident that the angle between the base and each of the slanted faces is 45 degrees. Thus, the right triangle in the picture above is an isosceles right triangle.

Now consider the two points, P and Q, where one of the top edges of the cube touches the pyramid. Let S be the corner of the cube directly below P and let R be the corner directly below Q. Extend the bottom edge RS to touch the sides of the pyramid’s base at T and U. Now the length of PQ is E as is the length of QR, PS and RS. They are all edges of the cube. But since the angle between each of the slanted faces and the base of the pyramid is 45 degrees, RT and SU are also of length E. Clearly, TU is of length 3E. Equally clear is that this must equal 2 units (the length of the sides of the base). Putting this all together yields

\[ E = \frac{2}{3}. \]

Congratulations to Andrew Scotti of Christ the King Regional High School! He was the sole correct solver for this problem.