The Mathematics Department
Presents
The Problem of the Month
April 2004
The Roads Not Taken

The Problem:
Imagine that you live in the city of Squaresville. The roads in Squaresville are all quite odd. The streets all run North-South and are one-way heading North. The avenues all run East-West and are one-way heading East. The street numbers increase from South to North. The avenue numbers increase from West to East. You live on the corner of 1st Avenue and 1st Street, the extreme South-West corner of town. You wish to drive to the intersection of 7th Street and 5th Avenue. Observing all of the one-way street regulations, how many different routes can you take from home to your destination? (Routes are considered different if they differ for even one block.)

A Solution:

At each intersection of street and avenue compute the number N(S, A).
Here “S” is the street number and “A” is the avenue number. N(S, A) is the number of different routes from your home at (1,1) to the intersection of Street S and Avenue A. Note that, for S and A greater than 1, this number can be defined recursively as follows:

\[ N(S, A) = N(S-1, A) + N(S, A-1) \]

The reason for this should be clear. The number of different routes to the intersection of S and A equals the number of different routes to (S-1, A) added to the number of different routes to (S, A-1). For all the intersections on 1st Street, this number has value 1. Thus \( N(1, A) = 1 \) for all values of A. Similarly, \( N(S, 1) = 1 \) for all values of S. Computing this value recursively yields \( N(7, 5) = 210 \).

One can also find a “closed form” solution. Graphing several values on a grid makes clear that the N(S, A) values are all values from Pascal’s Triangle.
N(5,1)=1    N(5,2)=5    N(5,3)=15    N(5,4)=35    N(5,5)=70
N(4,1)=1    N(4,2)=4    N(4,3)=10    N(4,4)=20    N(4,5)=35
N(3,1)=1    N(3,2)=3    N(3,3)=6     N(3,4)=10    N(3,5)=15
N(2,1)=1    N(2,2)=2    N(2,3)=3     N(2,4)=4     N(2,5)=5
N(1,1)=1    N(1,2)=1    N(1,3)=1     N(1,4)=1     N(1,5)=1

Observe that this is just Pascal’s Triangle rotated through 135 degrees. The diagonals from upper left to lower right are the rows of the triangle. A moment’s reflection yields:

\[ N(S, A) = \binom{S + A - 2}{S - 1} \]

By this we mean the binomial coefficient S+A-2, choose S-1.