The Mathematics Department
Presents
The Problem of the Month
September 2021

The Problem:

The simplest Archimedean spiral is given by the polar equation $r = \theta$ where θ is given in radians. Consider the unit square with all vertices of the form $(\pm 1/2, \pm 1/2)$. Clearly the spiral starts off inside the square at the origin but quickly grows so that it spirals around outside the square. As such, the spiral must intersect the square at a point, where it crosses from inside to outside. Find the coordinates of this point to two places of accuracy.

The Solution:

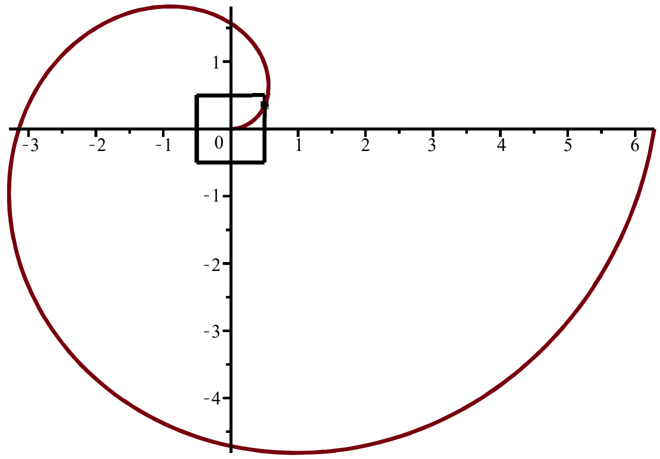
> $restart : with(plots) : with(plottools) :$

> $a := line\left(\left[-\frac{1}{2}, -\frac{1}{2}\right], \left[-\frac{1}{2}, \frac{1}{2}\right]\right) : b := line\left(\left[-\frac{1}{2}, -\frac{1}{2}\right], \left[\frac{1}{2}, \frac{-1}{2}\right]\right) :$

> $c := line\left(\left[\frac{1}{2}, \frac{1}{2}\right], \left[-\frac{1}{2}, \frac{1}{2}\right]\right) : dd := line\left(\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{-1}{2}\right]\right) :$

> $e := implicitplot(r = \theta, \theta = 0..2\cdot \mathrm{Pi}, r = 0..2\cdot \mathrm{Pi}, coords = \mathrm{polar}) : f := pointplot([.5, .3494],$
> $symbol = solidcircle, symbolsize = 10) :$

> $display([a, b, c, dd, e, f], scaling = constrained);$



> $evalf\left(\frac{\text{sqrt}(2)}{2}\right);$

$$0.7071067810 \quad \textbf{(1)}$$

> $evalf\left(\frac{\text{Pi}}{4}\right);$

$$0.7853981635 \quad \textbf{(2)}$$

>

Note that the vertex of the square in the first quadrant is $\frac{\sqrt{2}}{2} = 0.707$ units from the origin and the distance from the origin when $\theta = \frac{\pi}{4} = 0.785$. Thus, the spiral is outside the square when $\theta = \frac{\pi}{4} = 0.785$ so it must have crossed an edge of the square before this value of θ. So how do we find the crossing point?

> $restart : with(plots) : with(plottools) :$

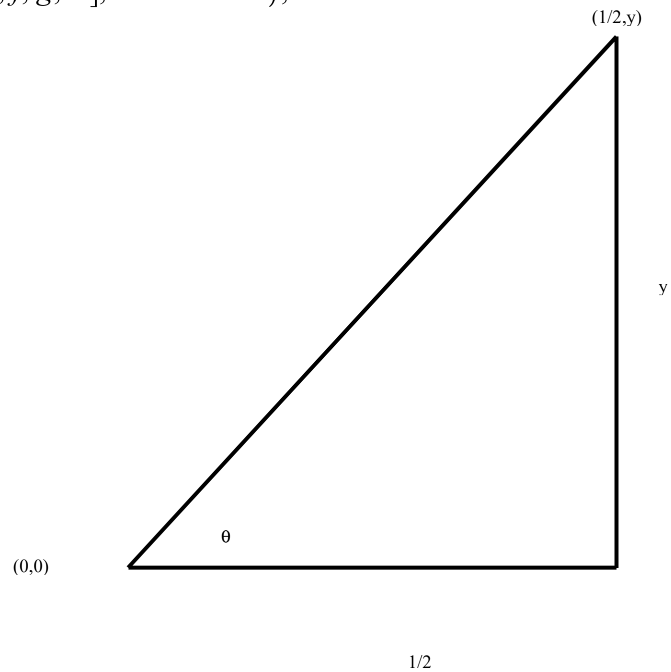
> $a := line\left(\left[\frac{1}{2}, 0\right], \left[\frac{1}{2}, .85\right]\right) : b := line\left(\left[\frac{1}{2}, .85\right], [0, 0]\right) : c := line\left([0, 0], \left[\frac{1}{2}, 0\right]\right) :$

> $dd := textplot([.1, .05, \text{"θ"}]) : e := textplot([.3, -.15, \text{"1/2"}]) : f := textplot([.55, .45, \text{"y"}]) :$

> $g := textplot([-.1, 0, \text{"(0,0)"}]) : h := textplot\left(\left[\frac{1}{2}, .88, \text{"(1/2,y)"}\right]\right) :$

>

> *display*([*a*, *b*, *c*, *dd*, *e*, *f*, *g*, *h*], *axes* = *none*);



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Above, we have drawn the right triangle with legs $\frac{1}{2}$ and y, the coordinate we seek. The hypotenuse is also of length θ since the point (1/2, y) is on the spiral. Thus, we have $\theta^2 = y^2 + \frac{1}{4} \Rightarrow y = \sqrt{\theta^2 - \frac{1}{4}}$.

We also have $\cos(\theta) = \frac{1}{2\,\theta}$

Let us use the first three terms of the Taylor series for cos:

$$\cos(\theta) = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4$$

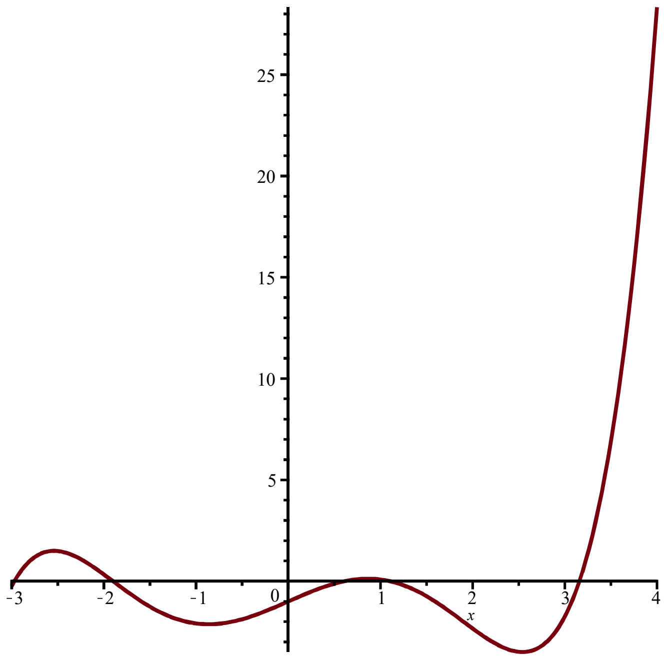
Let us take $\cos(\theta) = \frac{1}{2\,\theta}$, cross multiply and subtract 1 to get:

$$\frac{1}{12}\theta^5 - \theta^3 + 2\,\theta - 1 = 0$$

> *restart* : *with*(*plots*) : *with*(*plottools*) :

> $a := plot\left(\frac{1}{12}x^5 - x^3 + 2\,x - 1, x = -3\,..4\right)$:

> *display*([*a*]);



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Let use Newton's method.

For simplicity, let us use x in place of θ. The polynomial that we must find a root for is:

$$f(x) = \frac{x^5}{12} - x^3 + 2\,x - 1 = 0$$

Newton's method says: take an initial guess, x_1. Then compute:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Iterate this until $|x_n - x_{n+1}|$ is within the desired degree of accuracy.

The function has 5 real roots. Plugging approximations of four of them into $y = \sqrt{\theta^2 - \frac{1}{4}}$ show them to be unsuitable. The root just above 1/2 looks promising. A cursor probe tells us that is near 0.618. Let us take this to be our initial guess.

> $restart : with(plots) : with(plottools) :$

> $f := x \rightarrow \frac{1}{12}x^5 - x^3 + 2\,x - 1;$

$$f := x \mapsto \frac{1}{12} \cdot x^5 - x^3 + 2 \cdot x - 1 \quad \textbf{(3)}$$

> $fp := x \rightarrow \frac{5}{12}x^4 - 3\,x^2 + 2;$

$$fp := x \mapsto \frac{5}{12} \cdot x^4 - 3 \cdot x^2 + 2 \quad \textbf{(4)}$$

> $x1 := 0.618;$

$$x1 := 0.618 \quad \textbf{(5)}$$

> $x2 := x1 - \frac{f(x1)}{fp(x1)};$

$$x2 := 0.6098218369 \quad \textbf{(6)}$$

> $x3 := x2 - \frac{f(x2)}{fp(x2)};$

$$x3 := 0.6099391122 \quad \textbf{(7)}$$

>

OK, we can stop here since the last two values generated only differ in the 4th decimal position.

> $y := \mathrm{sqrt}(0.610^2 - .25);$

(8)

$$y := 0.3494281042 \tag{8}$$

>

Plugging this last value into $y=\sqrt{\theta^2-\frac{1}{4}}$ yields y=0.3494.

So the coordinates of the intersection of the square and the spiral are **(0.5, 0.3494)** to within two decimal places of accuracy.