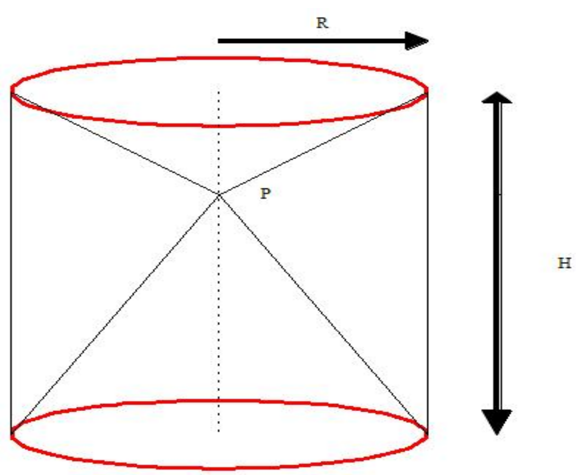
The Mathematics Department
Presents
The Problem of the Month
February 2024

The Problem:

Consider a cylinder of height H and radius R. Let P be an *arbitrary* point on the vertical axis of the cylinder. Construct two cones, one with base the top of the cylinder and one with base the bottom, both having P as a vertex. See the picture below. Let V be the volume contained *within* the cylinder and *outside* the two cones. Show that V is independent of the choice of P and find the value of V as a function of H and R.



The Solution:

The formula for the volume of a cylinder is 1/3(area of base) x the height. Let P be h above the base. Then the height of the bottom cone is h and the height of the top cone is H - h. Thus, the volume of the bottom cone is:

$$\frac{1}{3}\left(\pi R^2\right)h$$

Clearly, the volume of the top cone is:

$$\frac{1}{3}\left(\pi R^2\right)(H-h)$$

If we add these two volumes together, we get:

$$\frac{1}{3}\left(\pi R^2\right)[h+H-h]=\frac{1}{3}\left(\pi R^2 H\right)$$

Note that this is one third of the volume of the entire cylinder. Note, also, that it is independent of h. Thus, the choice of a particular point, P, was of no consequence.

As the volume contained in the two cones is one third the volume of the cylinder, the volume contained in the cylinder, outside the two cones will be two thirds of the volume of the cylinder. Thus it is:

$$\frac{2}{3}\left(\pi R^2 H\right)$$