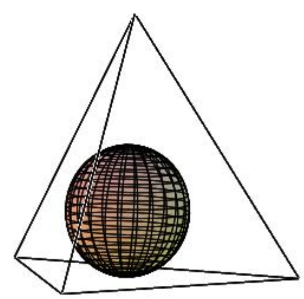
The Mathematics Department Presents The Problem of the Month March 2024

The Problem:

A regular tetrahedron is a polyhedron with four identical faces, each being an equilateral triangle. Imagine a sphere inscribed inside a regular tetrahedron. This is the largest sphere that can be enclosed within the tetrahedron. Suppose that each of six edges of our tetrahedron was of length s. Find the volume of the region contained within the tetrahedron but outside of the inscribed sphere. In other words, if the tetrahedron were hollow and the sphere were solid, how much air would there be inside the tetrahedron?



The Solution:

We begin with several observations:

1) A tetrahedron is a ***cone*** on a triangular base. A cone is a solid made by taking an enclosed region in the plane and connecting each point in the region to a single point (called the vertex) that does not lie in the plane. The volume of any cone is 1/3 times the product of the area of the base and the height.

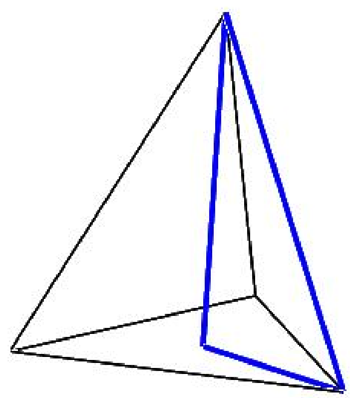
2) If each edge of our tetrahedron is of length s, then the area of each triangular face will be:

$$F = \frac{s}{2} \times \frac{\sqrt{3}}{2} s = \frac{\sqrt{3}}{4} s^2$$

3) The altitude of the tetrahedron is determined as follows. Take the point on the base equilateral triangle where the three angle bisectors meet. This will be the in-center of the base, i.e. the center of the inscribed circle in the base. The vertex of the tetrahedron will lie directly above this point. The distance from any vertex of the base to the in-center is:

$$\frac{2}{3}\frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{3}s$$

Recall that the in-center divides the altitude of an equilateral triangle in the ratio of 1:2. Thus the distance from any vertex to this point is two thirds of the altitude of the equilateral triangle. Now consider the right triangle with one leg the line segment connecting a vertex of the base to the in-center and the other leg being the altitude of the tetrahedron, *A*. The hypotenuse will be one of the edges of the tetrahedron. See the figure below:



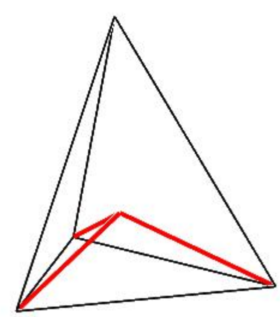
We have:

$$A^2 + \left(\frac{\sqrt{3}}{3}s\right)^2 = s^2 \rightarrow A^2 + \frac{1}{3}s^2 = s^2 \rightarrow A = s\sqrt{\frac{2}{3}}$$

Now we have the area of the base and the altitude of the tetrahedron. To get the volume, all we need to do is multiply these together and take 1/3 of the result. Thus, the volume of the tetrahedron is:

$$Vt = \frac{1}{3} \times \frac{\sqrt{3}}{4} s^2 \times s\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{12} s^3$$

4) Now for the sphere. In order to find the volume of the sphere, we need to know its radius. We determine this as follows. Consider a mini tetrahedron contained in the original tetrahedron with vertex the center of the inscribed sphere and base, one of the triangular faces of the larger tetrahedron. This mini tetrahedron will not be regular. See the figure below:



Now we could have made the identical construction using any of the four faces of the regular tetrahedron. Thus, four of these mini tetrahedrons precisely fill up the larger one. Consequently, the volume of this mini tetrahedron is one quarter of the volume of the larger. Observing that the altitude of the mini tetrahedron is the radius, *r*, of the inscribed sphere, we see that:

$$4 \times \frac{1}{3} rF = \frac{1}{3} AF \to r = \frac{A}{4} \to r = \frac{s}{4}\sqrt{\frac{2}{3}}$$

Now we are nearly done. Using the formula for the volume of a sphere $V = \frac{4}{3}\pi r^3$ we get:

$$Vs = \pi\sqrt{\frac{2}{3}}\frac{s^3}{72}$$

To complete the solution, all we need do is subtract the volume of the sphere from the volume of the tetrahedron to get:

$$\frac{\sqrt{2}}{12}\left(1-\frac{\pi}{6\sqrt{3}}\right)s^{3}$$