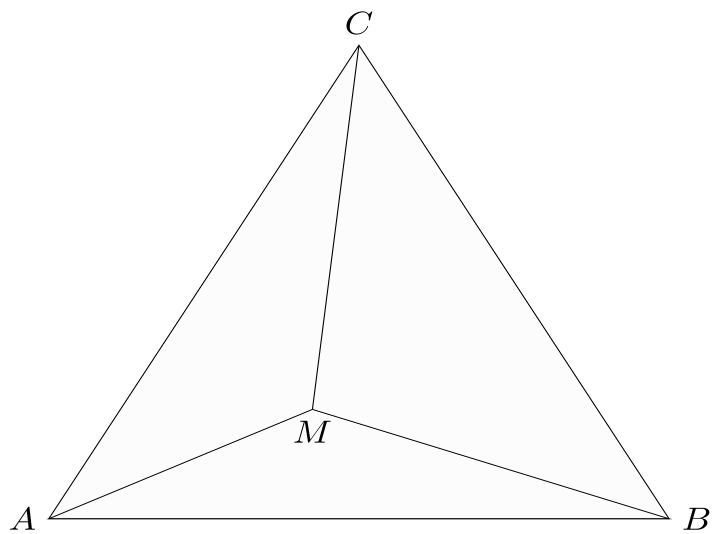
Solution to the Problem of the Month
March 2022

Consider an equilateral $\triangle ABC$ with each side being of length n. Let M be a point inside the triangle such that $|AM| = 3$, $|BM| = 4$ and $|CM| = 5$. See the sketch below. Find the value of n in the form $\sqrt{a + b\sqrt{c}}$, where a, b and c are integers.



A solution:

We begin by rotating $\triangle ABC$ in a counterclockwise direction $\pi/3$ radians about A. See the sketch below. Let M get mapped to M' and C to C', then $\triangle MM'A$ will form an equilateral triangle with sides of length 3. We now observe that $\triangle MM'C$ is a 3–4–5 triangle and it readily follows that $\measuredangle MM'C = \pi/2$ radians.

We can also see that $\measuredangle AM'C = \pi/2 + \pi/3 = 5\pi/6$. In $\triangle AM'C$, where $AC = n$, we apply the cosine rule to get

$$n = \sqrt{3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos(5\pi/6)} = \sqrt{25 + 12\sqrt{3}}.$$