Two solutions to the Problem of the Month for February 2023

Given a right triangle with legs *a* and *b* and hypotenuse *c*, prove that:

$$a + b \leq \sqrt{2}\, c$$

Under what conditions, do we have equality?

<u>Solution</u>:

$$a + b \leq \sqrt{2}\, c \Leftrightarrow \frac{(a+b)}{\sqrt{2}} \leq \sqrt{a^2 + b^2}$$

$$\Leftrightarrow \frac{\left(a^2 + b^2 + 2\,ab\right)}{2} \leq a^2 + b^2$$

$\Leftrightarrow ab \leq \frac{1}{2}\left(a^2 + b^2\right)$ This is true by the arithmetic mean, geometric mean inequality which says that $\sqrt{st} \leq \frac{(s+t)}{2}$ and equality holds precisely when $s = t$.

Observe that our last inequality relates the arithmetic and geometric means of a^2 & b^2.

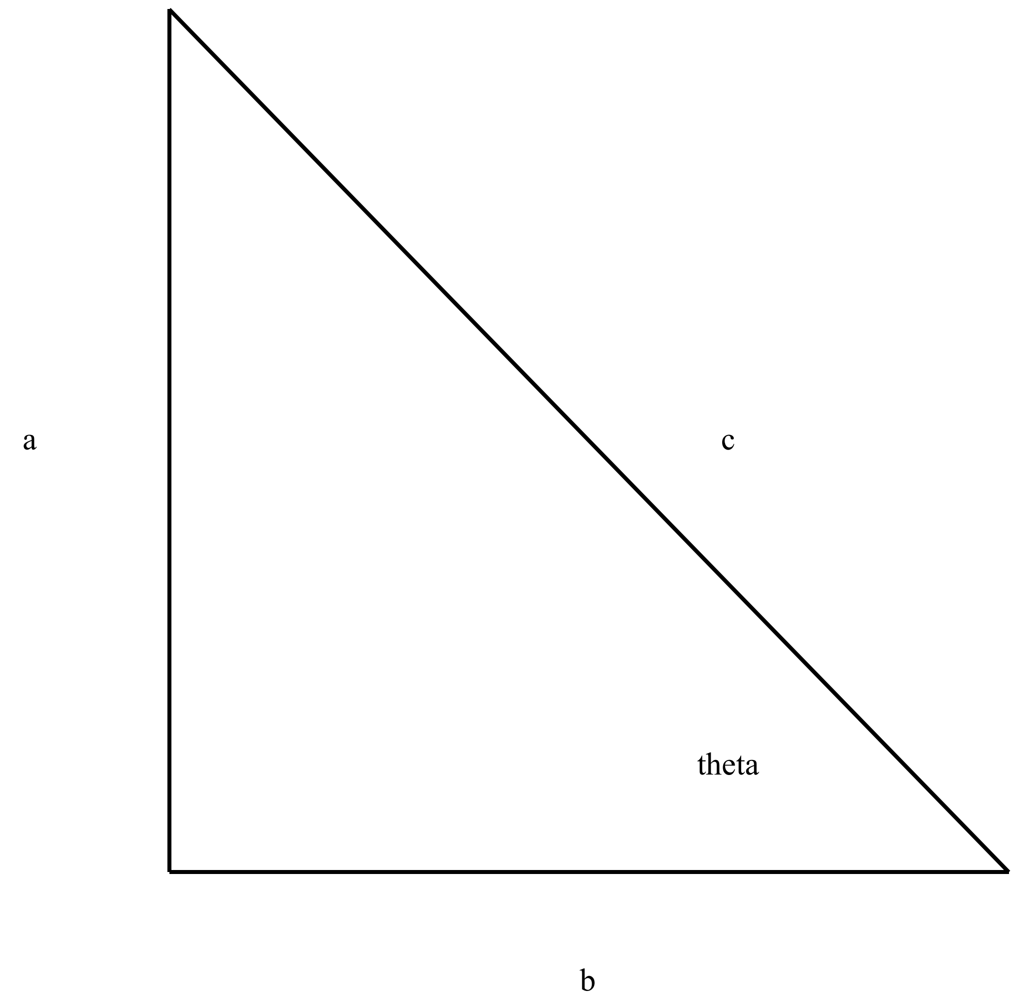
Thus we have equality precisely when the right triangle is isosceles.

****

****

Here is a second proof.

```
> restart : with(plots) : with(plottools) :
> a := line([0, 0], [3, 0]) : b := line([0, 0], [0, 4]) : c := line([3, 0], [0, 4]) :
> dd := textplot([-.5, 2, "a"]) : e := textplot([1.5, -.5, "b"]) : f := textplot([2, 2, "c"]) :
> g := textplot([2, .5, "theta"]) :
> display([a, b, c, dd, e, f, g], axes = none);
```



From the diagram above, we have

$b = c \cdot \cos(\text{theta}) \quad \& \quad a = c \cdot \sin(\text{theta})$

Thus

$$a + b = c(\cos(\text{theta}) + \sin(\text{theta})) = c\sqrt{2}\, \sin\left(\text{theta} + \frac{\pi}{4}\right)$$

$$\leq c\sqrt{2}$$

We have equality when theta = $\frac{\pi}{4}$ making the right triangle isosceles.

Here we recall the identity:

$$\cos(\text{theta}) + \sin(\text{theta}) = \sin\left(\text{theta} + \frac{\pi}{4}\right)$$

This identity is proved as follows:

$$\sin(t) + \cos(t) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(t) + \frac{1}{\sqrt{2}}\cos(t)\right)$$

$$= \sqrt{2}\cdot\left(\cos\left(\frac{\pi}{4}\right)\sin(t) + \sin\left(\frac{\pi}{4}\right)\cos(t)\right) = \sqrt{2}\sin\left(\frac{\pi}{4} + t\right)$$

This last identify comes from

$$\sin(a + b) = \cos(a)\sin(b) + \cos(b)\sin(a)$$