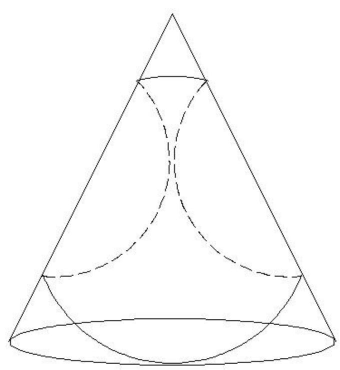
The Mathematics Department Presents The Problem of the Month February 2022

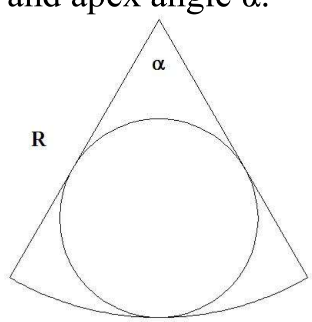
The Problem:

Imagine a right circular cone with altitude H and slant height R. Suppose that circular disk were to be wrapped around the cone like a blanket. Find r, the radius of the largest such disk that can be confined to the surface of the cone without over-lapping itself. Express r in terms of H and R. See diagram below:



The Solution:

Observe, first, that the radius of the circular base of the cone is $\sqrt{R^2 - H^2}$. This makes the circumference of the circular base $C = 2\pi\sqrt{R^2 - H^2}$. Now, imagine slicing the cone along the slant height line from apex to base that passes through the point where the circular disk is tangent to itself. Unfurl the sliced cone to get a sector of a circle with radius R and apex angle α.

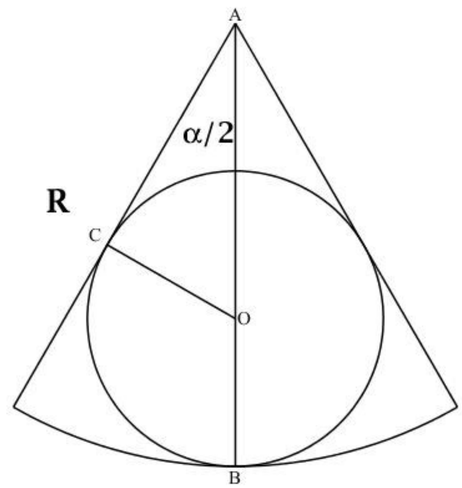


Now:

$$\frac{C}{2\pi R} = \frac{\alpha}{2\pi} \Rightarrow \alpha = \frac{2\pi\sqrt{R^2 - H^2}}{R}$$

Let us compute the radius of the inscribed circle in terms of *R* and α and then substitute the value above for α.

Consider the following diagram:



Draw a line segment from the apex of the sector, A, to the point, B, where the curved end of the sector is tangent to the inscribed circle. This segment will bisect the apex angle and it will pass through the center, O, of the inscribed circle. Draw segment OC from the center of the inscribed sector to a point, C, of tangency of the circle and the edge of the sector. Now |AO| = R-r and |OC| = r. Now sin(α/2) = r / (R-r).

Thus R sin(α/2) – r sin(α/2) = r

This yields R sin(α/2) = r(sin(α/2) + 1)

Solving for r yields the answer: r = R sin(α/2) / [1+ sin(α/2)]

(Note that this was done in the solution to problem 12-21.)

Now let us substitute our expression for α into this formula:

$$r = \frac{R\sin\left(\frac{\pi\sqrt{R^2 - H^2}}{R}\right)}{\left[1 + \sin\left(\frac{\pi\sqrt{R^2 - H^2}}{R}\right)\right]}$$