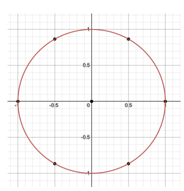
The Problem of the Month
November 2023 Solution

In a closed unit disk, seven points are located inside so that they are at least one unit apart from one another. Show that one of the points must be located at the center of the disk. Make a sketch of a unit disk and a configuration of the seven points that depict the scenario proposed by the problem.

Without loss of generality, partition the unit disk with its center, O removed, into six regions, described in polar form by $0 < r \leq 1$ in each case with $0 \leq \theta < \frac{\pi}{3}$, $\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}, \cdots$, $\frac{5\pi}{3} \leq \theta < 2\pi$. Assume that none of the points are at the center of the disk. It follows by the pigeonhole principle that one of the six sectors must contain at least two of the points. For any two points, P and Q in a sector, we have that $\measuredangle POQ < \frac{\pi}{3}$ and one can show that the distance between P and Q is smaller than 1. (See the remark below.)

For the sketch, without loss of generality (as we can readily rotate the unit circle), place one of the points at the center of the disk and the remaining six points at $(r, \theta) \in \{(1, 0), (1, \pi/3), (1, 2\pi/3), (1, \pi), (1, 4\pi/3), (1, 5\pi/3)\}$.



Remark: Let P and Q be any two points inside the sector described by $0 < r \leq 1$ and $0 \leq \theta < \frac{\pi}{3}$. Let $A = (1, 0)$ and $B = (1, \pi/3)$. Let the length of OP and OQ be denoted by $|OP| = m$ and $|OQ| = n$ respectively and let $\measuredangle POQ = \theta_1$ where $0 < \theta_1 < \frac{\pi}{3}$. It follows by the cosine rule that $|PQ|^2 = |OP|^2 + |OQ|^2 - 2|OP| \cdot |OQ| \cos\theta_1 = m^2 + n^2 - 2mn\cos\theta_1$.
Let $f(m, n) = m^2 + n^2 - 2mn\cos\theta_1$, and we get the partial derivatives, $f_m = 2m - 2n\cos\theta_1$ and $f_n = 2n - 2m\cos\theta_1$. Optimizing, give us $m = n$, where $0 < m, n < 1$. The optimal value for $|PQ|$ is $\sqrt{2m^2 - 2m^2\cos\theta_1} = 2m\sin\frac{\theta_1}{2} < 2 \cdot 1 \sin\frac{\pi}{6} = 1$.