The Mathematics Department
Presents
The Problem of the Month
November, 2022

The Problem:

Find integers k and m such that $1 < k < m$ and the sum of the integers 1 through k-1 equals the sum of the integers from k+1 up to and including m. That is, we seek the k and m such that the sum of the terms in each of the two bracketed expressions are equal.

(1 2 3 4 …. k-1) k (k+1 k+2 …. m)

For extra honors, show that the solution is not unique.

It would be helpful to recall the formula:

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

The Solution:

The sum of the numbers from 1 to $k-1$ is: $\frac{(k-1)\cdot k}{2}$.

The sum of the numbers from $k+1$ up to m is:

$$\frac{m\cdot(m+1)}{2} - \frac{k\cdot(k+1)}{2}.$$

Let us set these equal:

$$\frac{\left(m^2+m\right)}{2}=\frac{\left(k^2-k\right)}{2}+\frac{\left(k^2+k\right)}{2}$$

Thus: $m^2+m=2k^2$ so $k=\sqrt{\frac{m\cdot(m+1)}{2}}$

Since k must be an integer, $\frac{m\cdot(m+1)}{2}$ must be a perfect square.

One trivial way to get this is to set $m=\frac{(m+1)}{2}$. This gives $m = 1$.

Now, suppose we want a non-trivial solution. Let us make two observations. First, since m and $m+1$ are sequential, one must be even and the other odd. For the very same reason, these two numbers must have absolutely no primes in common in their prime decompositions. This means that if $\frac{m\cdot(m+1)}{2}$ is a perfect square, it must be the product of two perfect squares. There are two cases to consider.

<u>Case I:</u> m is even. Then

$$\frac{m}{2}=x^2 \Rightarrow m=2x^2 \Rightarrow m+1=2x^2+1=y^2.$$

Here, all of these variables must be integers. Note that y must be odd. Let us re-write this as $y^2-2x^2=1$. Clearly $y > 1$. Trying the next odd integer, $y = 3$, we hit pay-dirt.

Thus we have a solution in integers:

$$y^2-2x^2=1 \quad \Rightarrow 3^2-2\left(2^2\right)=9-8=1\ .$$

Thus $y=3, x=2, m=2\left(2^2\right)=8, m+1=9,$

$$k = \sqrt{\frac{m\cdot(m+1)}{2}} = \sqrt{\frac{8\cdot 9}{2}} = \sqrt{36} = 6 = k$$

This gives the solution:

1 2 3 4 5 **6** 7 8

$1 + 2 + 3 + 4 + 5 = 15 = 7 + 8.$

Case II: m+1 is even. All of the arguments follow as above except we get:

$2y^2 - x^2 = 1.$ If $x = 1$ then y =1 and m=1. This is the trivial solution.

Clearly x must be odd in this case. We find a solution when $x = 7$.

On the other hand, if $x = 7$, we get $y = 5$: $2\cdot 5^2 - 7^2 = 50 - 49 = 1.$
This makes $m = 49$ & $\frac{(m+1)}{2} = 25$

So $k = \sqrt{49\cdot 25} = 35.$

This gives:

1+2+3+...+34+**35**+36+....+49

Note that $\frac{34\cdot 35}{2} = 595$ and $\frac{49\cdot 50}{2} - \frac{35\cdot 36}{2} = 595$

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Note that these solutions are not unique. In both cases, m odd or even, we get multiple solutions. In the first case, below, we let y take on odd values from 1 to 17 and go in search of an integer solution to

$x = \sqrt{\frac{(y^2 - 1)}{2}}$. We find two non-trivial solutions. Had we gone beyond 17, we would have found more. In the second case, below, we let x take on the odd values up to 41 and go in search of integer solutions to $y = \sqrt{\frac{(x^2 + 1)}{2}}$. Again, we found two non-trivial solutions.

> *restart* :

> **for** y **from** 1 **to** 17 **by** 2 **do** $\mathrm{sqrt}\left(\frac{(y^2-1)}{2}\right)$ **end do**;

$$0$$
$$2$$
$$2\sqrt{3}$$
$$2\sqrt{6}$$
$$2\sqrt{10}$$
$$2\sqrt{15}$$
$$2\sqrt{21}$$
$$4\sqrt{7}$$
$$12 \qquad \textbf{(1)}$$

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> **for** x **from** 1 **to** 41 **by** 2 **do** $\mathrm{sqrt}\left(\frac{(x^2+1)}{2}\right)$ **end do**;

$$1$$
$$\sqrt{5}$$
$$\sqrt{13}$$
$$5$$
$$\sqrt{41}$$
$$\sqrt{61}$$
$$\sqrt{85}$$
$$\sqrt{113}$$

$$\sqrt{145}$$
$$\sqrt{181}$$
$$\sqrt{221}$$
$$\sqrt{265}$$
$$\sqrt{313}$$
$$\sqrt{365}$$
$$\sqrt{421}$$
$$\sqrt{481}$$
$$\sqrt{545}$$
$$\sqrt{613}$$
$$\sqrt{685}$$
$$\sqrt{761}$$
$$29 \tag{2}$$

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Note that in both cases, we are looking for points with integer coordinates on a hyperbola.

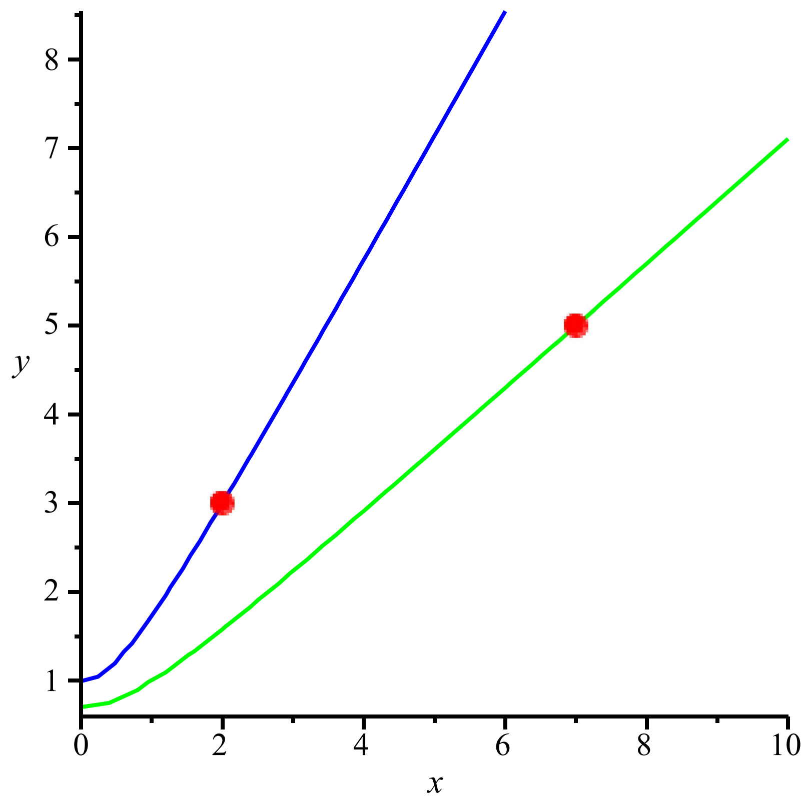
> $restart:$

> $with(plots): with(plottools):$

> $a := implicitplot\left(y^2 - 2\,x^2 = 1, x = 0..10, y = 0..10, color = blue\right) : c := pointplot([2, 3], symbol = solidcircle, color = red, symbolsize = 20):$

> $b := implicitplot\left(2\,y^2 - x^2 = 1, x = 0..10, y = 0..10, color = green\right) : dd := pointplot([7, 5], symbol = solidcircle, color = red, symbolsize = 20):$

> $display([a, b, c, dd]);$



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The interested reader might want to look up Pell's equation: $x^2 - Ny^2 = 1$ and related equations.