The Problem of the Month October 2022

The cube and the octahedron are dual Platonic solids in the following sense. Each has the same number of vertices as the other has faces. As such, it is possible to place a cube inside an octahedron so that each vertex of the cube lies at the in-center of a face of the octahedron. Likewise, one can place an octahedron inside a cube in a similar fashion. Start with a cube, inscribe it with an octahedron, as just described, and then inscribe this octahedron with a cube. Find the ratio of the lengths of the edges of the large cube to those of the small cube.

The Solution:

We approach this problem in the following way. We begin with an octahedron with each side being of length 1, as opposed to a variable such as *L*. We lose no generality in doing so, as we are looking for a ratio of lengths. Then, we derive the length of the edges of the inscribed cube. Finally, we circumscribe the octahedron with a cube and compare the lengths of the edges of the large and small cube.

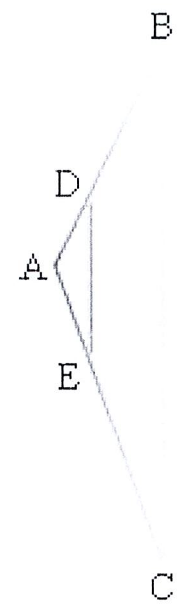
Let us find the length of the edges of a cube enclosed in an octahedron (with each edge of length one) such that each vertex of the cube touches the in-center of each triangular face of the octahedron.

Let us begin by collecting some information about the unit octahedron and the unit equilateral triangle.

First the altitude of the octahedron is $\sqrt{2}$. This is because the distance between antipodal vertices is the length of the diagonal of a unit square.

Next, in a unit equilateral triangle the distance from each vertex to the in-center is $\sqrt{3}/3$, the distance from the mid-point of each edge to the in-center is half this length, $\frac{\sqrt{3}}{6}$ and the altitude of the triangle is the sum of these two values, $\frac{\sqrt{3}}{2}$.

Now consider the in-centers of two adjacent faces of the octahedron. We need to determine the distance between them. Consider the following diagram:



Here, we are looking at the cross section of the left half of the octahedron. A is the midpoint of the left edge of the octahedron. B and C are the upper and lower vertices of the octahedron. D and E are the in-centers of the upper and lower left faces. Clearly triangles ABC and ADE are similar. Moreover, we know that:

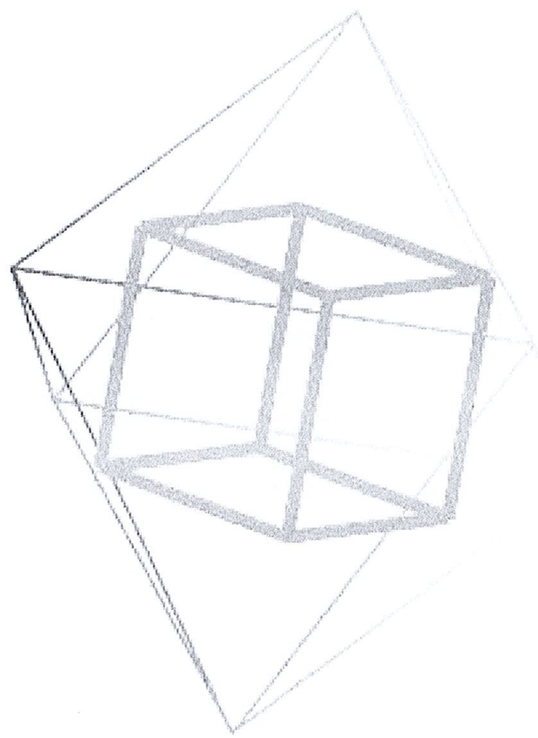
$|AB| = |AC| = \frac{\sqrt{3}}{2}$, $|BC| = \sqrt{2}$, $|AD| = |AE| = \frac{\sqrt{3}}{6}$ We seek $x = |DE|$. This will be the length of each edge of the cube.

By similar triangles, we have:

$$\frac{\frac{\sqrt{3}}{6}}{x} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{2}} \Rightarrow \frac{\sqrt{6}}{6} = \frac{\sqrt{3}}{2}x \Rightarrow \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{6}}{6} = \frac{\sqrt{2}}{3} = x.$$

So the length of each edge of the cube inside the octahedron will be $\frac{\sqrt{2}}{3}$.

Let us draw this:



Now, what about the outer cube. This is a triviality. Each edge of the octahedron is of length 1 and it connects the in-centers of two adjacent faces of the cube. In each case, a cross-section will be an isosceles right triangle with hypotenuse equal to 1. Hence the other two legs will be of length $\frac{\sqrt{2}}{2}$.

This makes the lengths of the edges of the outer cube

all equal to $\sqrt{2}$.

Putting this all together, we find that the ratio of the edge-length of the outer cube to that of the inner cube is 3 to 1.

Two observations:

First, as the inner cube has sides $\frac{\sqrt{2}}{3}$, we see that if we connect the in-centers of adjacent faces of this cube, we will get a right triangle with legs both of length $\frac{\sqrt{2}}{6}$. This makes the hypotenuse equal to 1/3. This will be the length of the sides of the octahedron inscribed inside this small cube. Thus, if we had gone from octahedron to cube to octahedron instead of cube to octahedron to cube, we would also have gotten a ratio of 3 to 1. Observe that this is to be expected as we are applying the same two scaling factors, just in different orders.

Secondly, we note that when one inscribes a tetrahedron, which is self-dual, inside a larger one using the same convention of placing each vertex at an in-center of a face, we also get a ratio of edge lengths of 3 to 1.