Problem of the Month for October 2021 Solution

Let us begin with a rectangular prism and replace its roof with a plane tilted at an angle (see picture below --- think of the Diamond Building in Chicago). That is, we take the plane that contains a line segment joining two opposite vertical edges of the prism along with one of the vertices atop a different vertical edge. We take the base to be 1 unit by 1 unit. Let us take the height of the prism to be H units, but we shall see that this height is irrelevant. (Below, we set H = 10.) We want to find a formula for the area of the slanted roof as a function of the angle of slant.

> $restart : with(plots) : with(plottools) : ang := \frac{\mathrm{Pi}}{3} : hh := \mathrm{sqrt}(2)\cdot\tan(ang) :$

> $a := line\left(\left[\frac{1}{2}, -\frac{1}{2}, 0\right], \left[\frac{1}{2}, \frac{1}{2}, 0\right]\right) : b := line\left(\left[\frac{1}{2}, \frac{1}{2}, 0\right], \left[-\frac{1}{2}, \frac{1}{2}, 0\right]\right) : c := line\left(\left[-\frac{1}{2}, \frac{1}{2}, 0\right], \left[-\frac{1}{2}, -\frac{1}{2}, 0\right]\right) : dd := line\left(\left[-\frac{1}{2}, -\frac{1}{2}, 0\right], \left[\frac{1}{2}, -\frac{1}{2}, 0\right]\right) :$

> $e := line\left(\left[\frac{1}{2}, -\frac{1}{2}, 0\right], \left[\frac{1}{2}, -\frac{1}{2}, 10\right]\right) : f := line\left(\left[\frac{1}{2}, \frac{1}{2}, 0\right], \left[\frac{1}{2}, \frac{1}{2}, 10\right]\right) : g := line\left(\left[-\frac{1}{2}, \frac{1}{2}, 0\right], \left[-\frac{1}{2}, \frac{1}{2}, 10\right]\right) : h := line\left(\left[-\frac{1}{2}, -\frac{1}{2}, 0\right], \left[-\frac{1}{2}, -\frac{1}{2}, 10\right]\right) :$

> $i := line\left(\left[\frac{1}{2}, -\frac{1}{2}, 10\right], \left[\frac{1}{2}, \frac{1}{2}, 10\right]\right) : j := line\left(\left[\frac{1}{2}, \frac{1}{2}, 10\right], \left[-\frac{1}{2}, \frac{1}{2}, 10\right]\right) : k := line\left(\left[-\frac{1}{2}, \frac{1}{2}, 10\right], \left[-\frac{1}{2}, -\frac{1}{2}, 10\right]\right) : l := line\left(\left[-\frac{1}{2}, -\frac{1}{2}, 10\right], \left[\frac{1}{2}, -\frac{1}{2}, 10\right]\right) :$

> $m := implicitplot3d\left(\frac{hh}{2}\cdot\left(x + \frac{1}{2}\right) + \frac{hh}{2}\cdot\left(y + \frac{1}{2}\right) + (z - 10) = 0, x = -\frac{1}{2} .. \frac{1}{2}, y = -\frac{1}{2} .. \frac{1}{2}, z = 0 .. 10\right) :$

> $n := pointplot3d\left(\left[\left[-\frac{1}{2}, -\frac{1}{2}, 10\right], \left[\frac{1}{2}, \frac{1}{2}, 10 - hh\right], \left[\frac{1}{2}, -\frac{1}{2}, 10\right], \left[-\frac{1}{2}, \frac{1}{2}, 10\right]\right], symbol = solidcircle, symbolsize = 15, color = black\right) :$

> $nn := pointplot3d\left(\left[\left[\frac{1}{2}, -\frac{1}{2}, 10 - \frac{hh}{2}\right], \left[-\frac{1}{2}, \frac{1}{2}, 10 - \frac{hh}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, 10 - hh\right]\right], symbol$

$= solidcircle, symbolsize = 15, color = black\Big):$

> $o := textplot3d\left(\left[-\frac{1}{2}, -\frac{1}{2}, 10.4, "A"\right]\right) : p := textplot3d\left(\left[\frac{1}{2}, \frac{1}{2}, 10 - 1.5 \cdot hh, "E"\right]\right):$
$q := textplot3d\left(\left[0.6, -\frac{1}{2}, 10, "C"\right]\right):$

> $r := textplot3d\left(\left[-\frac{1}{2}, 0.6, 10, "B"\right]\right) : s := textplot3d\left(\left[\frac{1}{2}, \frac{1}{2}, 10.4, "D"\right]\right) : t :=$
$textplot3d\left(\left[\frac{1}{2}, -0.6, 10 - \frac{hh}{2}, "F"\right]\right) : u := textplot3d\left(\left[-\frac{1}{2}, 0.6, 10 - \frac{hh}{2}, "G"\right]\right):$

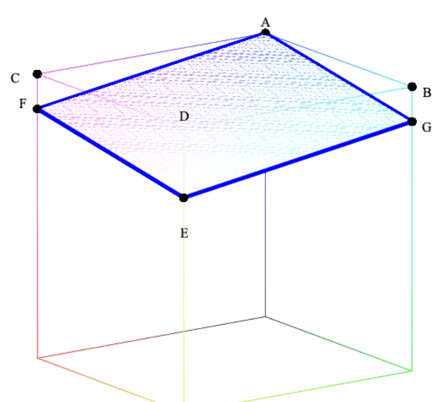
> $v := line\left(\left[-\frac{1}{2}, -\frac{1}{2}, 10\right], \left[\frac{1}{2}, \frac{-1}{2}, 10 - \frac{hh}{2}\right], thickness = 5, color = blue\right) : w := line\left(\left[\frac{1}{2},\right.\right.$
$\left.\left. -\frac{1}{2}, 10 - \frac{hh}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, 10 - hh\right], thickness = 5, color = blue\right):$

> $vv := line\left(\left[\frac{-1}{2}, \frac{1}{2}, 10 - \frac{hh}{2}\right], \left[\frac{1}{2}, \frac{1}{2}, 10 - hh\right], thickness = 5, color = blue\right) : ww :=$
$line\left(\left[\frac{-1}{2}, \frac{1}{2}, 10 - \frac{hh}{2}\right], \left[\frac{-1}{2}, \frac{-1}{2}, 10\right], thickness = 5, color = blue\right):$

>

>

> $display([a, b, c, dd, e, f, g, h, i, j, k, l, m, n, nn, o, p, q, r, s, t, u, v, vv, w, ww], axes = none);$



Here we have taken the following coordinates for the following points:

$A = \left(-\frac{1}{2}, -\frac{1}{2}, 10 \right)$ $B = \left(-\frac{1}{2}, \frac{1}{2}, 10 \right)$ $C = \left(\frac{1}{2}, -\frac{1}{2}, 10 \right)$ $D = \left(\frac{1}{2}, \frac{1}{2}, 10 \right)$ $E = \left(\frac{1}{2}, \frac{1}{2}, 10 - hh \right)$.

We seek the area of the slanted roof as a function of the angle of declination, *ang*. Consider the diagram, below. Here we have drawn triangle ADE. Note that $|AD| = \sqrt{2}$. Let $|DE| = hh$ and angle DAE be *ang*. Then we have:

$$hh = \sqrt{2} \cdot \tan(ang)$$

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> restart : with(plots) : with(plottools) :
> a := line([0, 0], [sqrt(2), 0]) : b := line([sqrt(2), 0], [sqrt(2), -1]) : c := line([0, 0],
      [sqrt(2), -1]) :
> dd := textplot([0, 0, "A"], align = {above}) : e := textplot([sqrt(2), 0, "D"], align
      = {above}) : f := textplot([sqrt(2), -1, "E"], align = {below}) :
> display([a, b, c, dd, e, f], axes = none);
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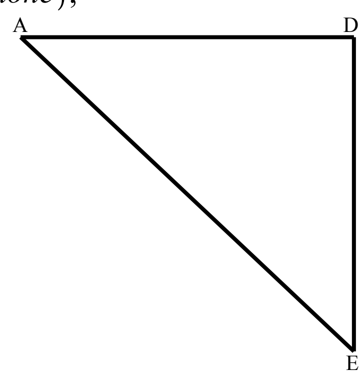

Now let us find the equation for the plane which represents the slanted roof. Clearly vector $\overrightarrow{AE} = \langle 1, 1, -hh \rangle$. We seek a vector orthogonal to this vector, as it will serve as the normal vector to our plane. An obvious choice is $N = \left\langle \frac{hh}{2}, \frac{hh}{2}, 1 \right\rangle$. (Note that the angle N makes with the z-axis equals ang.) If we take point A as the chosen point in the plane, we get equation:

$$PL: \quad \frac{hh}{2}\left(x+\frac{1}{2}\right)+\frac{hh}{2}\left(y+\frac{1}{2}\right)+(z-10)=0$$

Setting $x=\frac{1}{2}$ & $y=-\frac{1}{2}$, in the equation of our plane we get $z=10-\frac{hh}{2}$. So $F=\left(\frac{1}{2},-\frac{1}{2},\ 10-\frac{hh}{2}\right)$.

Likewise we set $x=-\frac{1}{2}$ & $y=\frac{1}{2}$ to get $G=\left(-\frac{1}{2},\ \frac{1}{2},\ 10-\frac{hh}{2}\right)$.

Now, let us compute the following vectors:

$$\overrightarrow{AF}=\left\langle 1,\ 0,-\frac{hh}{2}\right\rangle \quad \overrightarrow{AG}=\left\langle 0,\ 1,-\frac{hh}{2}\right\rangle \quad \overrightarrow{FE}=\left\langle 0,\ 1,-\frac{hh}{2}\right\rangle$$

$$\overrightarrow{GE}=\left\langle 1,\ 0,-\frac{hh}{2}\right\rangle$$

These vectors all have the same length: $\sqrt{1+\frac{hh^2}{4}}$

Using the formula $\frac{u \bullet v}{\|u\|\ \|v\|}=\cos(\theta)$ we see that angles GAF and GEF are congruent as are angles AFE and AGE. Thus, we have a rhombus.

The formula for the area of a rhombus is 1/2 the product of the diagonals. Let apply this:

$$|AE| = \sqrt{\left(-\frac{1}{2} - \frac{1}{2}\right)^2 + \left(-\frac{1}{2} - \frac{1}{2}\right)^2 + (10 - (10 - hh))^2}$$

$$= \sqrt{1 + 1 + hh^2} = \sqrt{2 + hh^2}$$

$$|FG| = \sqrt{\left(\frac{1}{2} + \frac{1}{2}\right)^2 + \left(-\frac{1}{2} - \frac{1}{2}\right)^2 + 0} = \sqrt{2}$$

Thus the area is

$$\frac{1}{2}\sqrt{2}\cdot\sqrt{2 + hh^2} = \frac{\sqrt{4 + hh^2}}{2} = \frac{\sqrt{4 + 4\cdot\tan^2(ang)}}{2}$$

$$= \sqrt{1 + \tan^2(ang)} = \frac{\mathbf{1}}{\mathbf{cos}(\boldsymbol{ang})} \mathbf{= sec}(\boldsymbol{ang})$$

Note that the height of the prism does not figure in the answer.