The Problem of the Month
April 2022

Let $f(x,y) = \sqrt{(x-1)^2+(y+1)^2} + \sqrt{(x+1)^2+(y-1)^2} + \sqrt{(x-2)^2+(y-2)^2}$.

Find the minimum value of $f(x,y)$ over the real numbers. Give your answer in the form $a\sqrt{b}+\sqrt{c}$, where a, b and c are positive integers.

A Solution.

We need to minimize the sum of the distances from the point (x,y) to the points $(1,-1)$, $(-1,1)$ and $(2,2)$. This is equivalent to finding the Fermat point of the triangle with vertices at $(1,-1)$, $(-1,1)$ and $(2,2)$. By symmetry, we see that this point must lie on the line $y=x$, where $0<x<2$. In light of this observation, we will minimize the function $m(x) = f(x,x)$, on the interval, $0<x<2$.

$$\begin{aligned} m(x) &= \sqrt{(x-1)^2+(x+1)^2} + \sqrt{(x+1)^2+(x-1)^2} + \sqrt{(x-2)^2+(x-2)^2} \\ &= 2\sqrt{2}\sqrt{x^2+1} + \sqrt{2}(2-x). \end{aligned}$$

The first derivative, $m'(x) = \frac{2\sqrt{2}x}{\sqrt{x^2+1}} - \sqrt{2}$, and when $m'(x)=0$, we see that the minimum value occurs at $x = \frac{1}{\sqrt{3}}$. It follows that $f(x,y)$ is minimized at the Fermat point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. The minimum value of $f(x,y)$ is

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = m\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{2}+\sqrt{6}.$$