Groups, Loops and KenKen

Groups are simple algebraic objects that arise naturally in a variety of settings. Here we examine the relationship between groups and the popular puzzle called KenKen.

First some definitions. A group is a set, G, that is endowed with a binary operation, *. The operation is a rule that takes as input two elements of G and generates an output element. We demand that, for every pair of elements from G, the output is also an element of G. Furthermore, we require that there exists a special element, e, in G with the property that $e * a = a * e = a$ for every a in G. This element, e, is called an identity element. As it can easily be shown to be unique, if it exists, we shall call it ***the*** identity element. Next, we ask that, for each element a in G there is an element a' in G with the property that

$a * a' = a' * a = e$. We call a' an inverse for a and, as it can be shown to be unique, if it exists, we refer to it as ***the*** inverse of a.

Finally, we note that * is a *binary* operation, meaning that it operates on pairs of elements of G. This leaves the value of $a * b * c$ to be ambiguous. Are we to interpret this as $(a * b) * c$ or $a * (b * c)$? We resolve this issue by setting the two expressions equal for all a, b and c. An operation with this last property is called associative.

Some, but not all, groups have the additional property that $a * b = b * a$ for all a and b in G. When this property holds, we call the group an Abelian group in honor of the Norwegian mathematician, Niels Abel.

A group can be infinite, like the integers under addition, or it can be as small as a single element G = $\{e\}$.

Small groups are often written out in tabular form. If G has n elements, then an n x n grid is generated with each of the columns labeled with a different element of G and, likewise, each of the rows. For example, if G= {*e, a, b, c*} where *e* is the identity element, we could write it out as follows:

*	***e***	***a***	***b***	***c***
e	*e*	*a*	*b*	*c*
a	*a*	*b*	*c*	*e*
b	*b*	*c*	*e*	*a*
c	*c*	*e*	*a*	*b*

Figure 1

The element in the box that lies at the intersection of a given row and column is the value of the row label * the column label. So, for instance, in the group depicted above, $a * b = c$.

For any finite group, the facts that there must be an identity element and that every element has a unique inverse, force the table to be a Latin square. That is, each element of G appears exactly once in each row and each column. Note that associativity of the operation was not needed to force a finite group table to be a Latin square.

If one takes all of the requirements in the definition of a group and removes the demand that the binary operation be associative, the resulting algebraic object one gets is called a loop. By the argument of the previous paragraph, we see that a finite loop table must also be a Latin square.

Sometimes, associativity arises naturally without being demanded. In particular, every loop with fewer than 5 elements is also a group. That is, in each of these cases, associativity “comes for free.”

Not only must every finite loop table be a Latin square, conversely, every Latin square gives rise to a loop table. We see this as follows. Begin with a Latin square. Label each column with a copy of the elements of the top row of the square. Next, label each row with a copy of the elements in the left-most column. The rows can be rearranged so that the ordering of the labels of the rows and columns is the same but this is not really necessary. The information in the table is the same whether or not we do this. We illustrate this below.

Start with the following 4 x 4 Latin square:

4	1	3	2
2	4	1	3
1	3	2	4
3	2	4	1

Figure 2

Next, we label the rows and columns as discussed and we permute the rows so that the order of the column headings and the row headings match.

*	**4**	**1**	**3**	**2**
4	4	1	3	2
1	1	3	2	4
3	3	2	4	1
2	2	4	1	3

Figure 3

Now, take the element in the upper left cell, 4 in this case, and observe that it plays the role of the identity. Moreover, as this element appears exactly once in each row and column, every element must have an inverse. Thus, we have generated a loop table.

Both loops and groups are algebraic structures that are independent of the symbols used to represent the elements. If the elements of a loop or a group were renamed, the structure would remain the same. In such a case, we say that the original object and the one that results from a renaming of the elements are isomorphic. For example, if we were to rename the elements 4, 1, 3 and 2 by *e, a, b* and *c*, respectively, then the table in Figure 3 would be identical to the that of Figure 1.

As such we see that the loop generated by the Latin square given in Figure 2 is, in fact a group. This is not a coincidence since, as we asserted above, every loop with fewer than 5 elements is also a group.

Now, it turns out that there are, up to isomorphism, exactly two distinct groups with 4 elements, both of which are Abelian groups. One is $Z_4 = \{0, 1, 2, 3\}$ with the operation * being addition modulo 4. Its group table is given below.

+	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Figure 4

Note that this group is isomorphic to the one given in Figure 1, the renaming replaces 0, 1, 2 and 3 by *e, a, b* and *c*, respectively.

The other group with 4 elements is called the Klein 4-group and it is denoted by V. Its elements are traditionally labeled *e, a, b* and *c* and its operation is given in the table below.

*	e	a	b	c
e	*e*	*a*	*b*	*c*
a	*a*	*e*	*c*	*b*
b	*b*	*c*	*e*	*a*
c	*c*	*b*	*a*	*e*

Figure 5

Putting this all together, we see that every 4 x 4 Latin square gives rise to one of these two groups, up to isomorphism.

Enter KenKen. This is a popular number puzzle. It consists of a square grid (we restrict our attention to 4 x 4 KenKen) with certain sub-regions outlined by bold lines. In each of these regions, there will be an integer along with one of the four basic arithmetic operations: addition, subtraction, multiplication or division. The task of the puzzle solver is to place a number from the set {1, 2, 3, 4} in each cell so that the arithmetic operation applied to the numbers inserted into the cells in the region match the given integer for that region. Additionally, the solved puzzle must be a Latin square. Here is the unsolved KenKen that appeared in the *New York Times* on 7/15/2020 along with its solution.

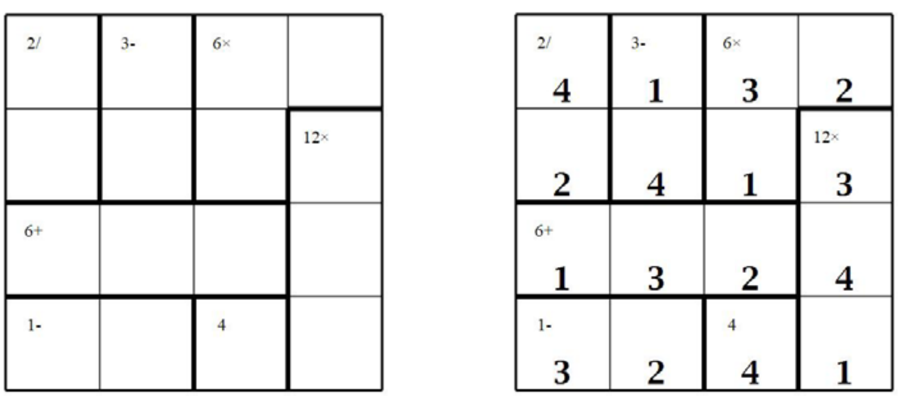

Figure 6

Observe that the solved KenKen in Figure 6 is just the Latin square given in Figure 2. Consequently, we see that this KenKen puzzle is isomorphic to Z_4.

As every 4 x 4 KenKen puzzle must be isomorphic to either Z_4 or V, we can exploit the properties of these two groups to help solve it. For example, both groups are Abelian. Thus, knowing that the entry in the row headed by 3 (row 4) and the column headed by 1 (column 2) must be a 2, we can deduce that the entry in the row headed by 1 (row 2) and the column headed by 3 (column 3) must also be a 2. Additionally, we since every element in V is its own inverse while Z_4 has only two entries that are their own inverse, we can deduce that if we find three entries that are their own inverses, the remaining entry must also be its own inverse.

At the outset, we said that groups arise naturally in a variety of settings. Here we see that your daily newspaper may be one of those settings.